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An alternative functional form for estimating the Lorenz curve

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ABSTRACT

We propose a simple single parameter functional form for the Lorenz curve. The new specification is fitted to existing data sets and is shown to provide a better fit than existing single parameter Lorenz curves for the given data.

1. Introduction

The Lorenz curve is an intuitive method for representing the distribution of income. Created by plotting cumulative income shares against cumulative population shares, the Lorenz curve forms the backbone of several inequality measures including the popular Gini coefficient. Lorenz curves may be constructed from grouped data using interpolation techniques (Gastwirth and Glauberman, 1976) or may be presumed to follow a particular parametric form and fitted to tabulated data (see Kakwani and Podder, 1976; Rasche et al., 1980; Ortega et al., 1991). Parametric forms such as these are advantaged over Lorenz curves constructed directly from grouped data as they do not assume homogeneity of incomes within subgroups and thus are not downwardly biased (Lerman and Yitzhaki, 1989). These techniques however face the disadvantage of imposing a rigid and in some cases unrealistic distribution upon the data and may result in poorly fitting Lorenz curves and inaccurate inequality estimates.

In this paper we propose an alternative single parameter functional form for the Lorenz curve and derive the implicit probability density function (PDF) and cumulative distribution function (CDF). We argue that these functions take appropriate shapes for modeling the distribution of income and demonstrate this by showing that the proposed method provides a better fit than other comparable Lorenz curves for our data set.

2. Fundamentals of the Lorenz curve

A Lorenz curve may be defined as

$$\eta = f(\pi) \tag{1}$$

where

- π is the cumulative population share of persons earning income equal to or below income level x.
- η is the cumulative income share of population subgroup π.

A Lorenz curve must have the following properties:

$$\frac{d\eta}{d\pi} > 0, \quad \frac{d^2\eta}{d\pi^2} > 0, \quad \eta(0) = 0, \quad \eta(1) = 1$$

and is defined on the domain $0 \le \pi \le 1$.

The most popular single parameter Lorenz curves are the forms proposed by Kakwani and Podder (1973), Gupta (1984) Chotikapanich (1993) and a form implied by the Pareto distribution. These are:

$$\text{Kakwani–Podder}: \eta(\pi) = \pi e^{-\delta(1-\pi)}, \delta > 0 \tag{2}$$

$$\text{Gupta}: \quad \eta(\pi) = \pi a^{\pi-1}, a > 1 \tag{3}$$

$$\text{Chotikapanich}: \quad \eta(\pi) = \frac{e^{k\pi} - 1}{e^k - 1}, k > 0 \tag{4}$$

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$$\text{Pareto}: \quad \eta(\pi) = 1-(1-\pi)^{\frac{1}{\gamma}}, \gamma>1 \tag{5}$$

Here we propose the functional form

$$\eta(\pi) = \pi\left(\frac{\beta-1}{\beta-\pi}\right), \beta>1 \tag{6}$$

It is simple to verify that Eq. (6) passes through the coordinate points (0,0) and (1,1) and that the first and second derivatives are greater than zero. The derivatives are

$$\frac{d\eta}{d\pi} = \pi\left(\frac{\beta-1}{(\beta-\pi)^2}\right) + \frac{\beta-1}{\beta-\pi} > 0 \text{ for } \beta>1, 0\le\pi\le 1 \tag{7}$$

$$\frac{d^2\eta}{d\pi^2} = 2\pi\left(\frac{\beta-1}{(\beta-\pi)^3}\right) + 2\frac{(\beta-1)}{(\beta-\pi)^2} > 0 \text{ for } \beta>1, 0\le\pi\le 1 \tag{8}$$

and are thus consistent with the required conditions.

A primary motivation for fitting a Lorenz curve is to facilitate the construction of inequality estimates such as the Gini coefficient. This widely used index may be defined as one less twice the area under the Lorenz curve. If the new functional form is used the Gini coefficient can be given as

$$G = 2\beta\left[(\beta-1)\ln\left(\frac{\beta-1}{\beta}\right)+1\right]-1 \tag{9}$$

such that the measure may be calculated directly in terms of parameter β.

Furthermore we can derive the implicit PDF and CDF for this Lorenz curve in terms of β and mean income level μ. Using Kakwani's (1980) result $x(F) = \frac{\eta'(\pi)}{\mu}$ the probability density function in terms of income (x) is

$$f(x;\alpha) = \frac{\alpha}{2x^2\sqrt{\frac{\alpha}{x}}} \quad \text{for} \quad \frac{\alpha}{\beta^2} \le x \le \frac{\alpha}{(\beta-1)^2} \tag{10}$$

where parameter $\alpha=\beta(\beta-1)\mu$.

The cumulative distribution function is

$$F(x) = \beta - \sqrt{\frac{\alpha}{x}} \quad \text{for} \quad \frac{\alpha}{\beta^2} \le x \le \frac{\alpha}{(\beta-1)^2} \tag{11}$$

The PDF of income defined in Eq. (10) has some unusual properties. Ostensibly this function of income is manipulated by the single parameter α, which depends on mean income level μ and Lorenz curve parameter β. Such a parameterization appears flawed as different combinations of β and μ can yield the same curve for Eq. (10). This does not imply that density functions with the same value for parameter α will be identical however, as the PDF is only defined on the domain ($x_{min} \le x \le x_{max}$) where the lower and upper bounds depend on both β and μ. As such each combination of β and μ defines a unique PDF for x, which is typically downward sloping over the domain in a manner similar to an exponential decay function.

Lorenz curves such as that proposed in Eq(6) which imply probability distribution functions that only exist on a subset of x are not uncommon, with the Chotikapanich, Gupta and Kakwani–Podder functional forms also exhibiting positive lower bounds and finite upper bounds. Such restrictions on x need not be unrealistic as institutional structures such as social welfare systems, minimum wages and high marginal tax rates may effectively constrain incomes to lie within certain bounds.

A further interesting property of the domain of Eq. (10) is that the distribution mean may be calculated directly from the upper and lower limits. The bounds on x are:

$$x_{min} = \frac{\alpha}{\beta^2} \quad x_{max} = \frac{\alpha}{(\beta-1)^2}$$

Solving these expressions with the equation for α gives the result

$$\mu = \sqrt{x_{min}x_{max}} \tag{12}$$

— that the mean of the implicit PDF is equal to the geometric average of the highest and lowest incomes available under the distribution.

Table 1
A comparison of functional forms for the Lorenz curve.

Data set	Kakwani–Podder		Chotikapanich		Pareto		Proposed	
	δ	MSE	k	MSE	γ	MSE	β	MSE
HK88	2.312	0.00196	3.302	0.00165	2.719	0.00068	1.282	**0.00044**
HK93	2.703	0.00270	3.737	0.00233	3.039	**0.00058**	1.230	0.00062
JP88	1.031	0.00027	1.700	0.00021	1.721	0.00060	1.774	**0.00009**
JP93	1.064	0.00036	1.747	0.00029	1.749	0.00052	1.743	**0.00012**
KR88	1.378	0.00068	2.171	0.00056	1.987	0.00067	1.542	**0.00023**
KR93	1.219	0.00025	1.957	0.00020	1.852	0.00101	1.637	**0.00011**
ML88	2.206	0.00138	3.180	0.00112	2.620	0.00096	1.301	**0.00024**
ML93	2.334	0.00148	3.326	0.00121	2.720	0.00101	1.280	**0.00025**
PH88	1.216	0.00057	1.956	0.00046	1.865	0.00047	1.631	**0.00015**
PH93	1.974	0.00115	2.910	0.00092	2.438	0.00090	1.347	**0.00020**
SG88	1.117	0.00033	1.820	0.00025	1.785	0.00060	1.702	**0.00008**
SG93	1.939	0.00098	2.868	0.00077	2.407	0.00104	1.356	**0.00017**
TW88	1.136	0.00045	1.847	0.00036	1.804	0.00051	1.686	**0.00015**
TW93	1.149	0.00040	1.864	0.00032	1.811	0.00058	1.678	**0.00012**
TL88	2.023	0.00150	2.968	0.00123	2.486	0.00068	1.334	**0.00031**
TL93	2.298	0.00176	3.286	0.00145	2.700	0.00075	1.285	**0.00032**

3. A comparison of single parameter functional forms

In this section we estimate the single parameter Lorenz curves given in Section (2) using decile data from Chotikapanich et al. (2007). The data covers income statistics from eight Asian countries in 1988 and 1993 and is referred to in an abbreviated form in Table 1. The abbreviations are: Hong Kong — HK, Japan — JP, Korea — KR, Malaysia — ML, Philippines — PH, Singapore — SG, Taiwan — TW and Thailand — TL. Each Lorenz curve is fitted to every data set and we follow Sarabia et al. (1999) by measuring the goodness of fit with the Mean Squared Error. We calculate this as

$$\text{MSE} = \sum_{i=1}^{n}\frac{(\eta_i-\eta(\pi_i))^2}{n} \tag{13}$$

where η_i is the cumulative income share of population group i calculated from raw data and $\eta(\pi_i)$ is the fitted value of the Lorenz curve at π_i. We exclude the Gupta Lorenz curve from this analysis as it can be shown to be functionally equivalent to the Kakwani–Podder specification and thus gives identical goodness of fit statistics[2]. The results are presented in Table 1, with parameter estimates and goodness of fit statistics given for each Lorenz curve.

The results demonstrate the capability of the proposed functional form to closely model a variety of different data sets. The new Lorenz curve provides the best fit (as measured by MSE) of the four single parameter forms in 15 of the 16 considered cases (indicated on the table in bold) with the Pareto Lorenz curve being slightly superior for Hong Kong data in 1993. The Chotikapanich and Pareto specifications appear roughly equivalent at fitting the given data while the Kakwani–Podder functional form was the poorest performer.

[2] Setting $\delta=\ln a$ allows the Kakwani-Podder specification given in Eq. (2) to be written in terms of the Gupta specification in Eq. (3).

4. Conclusion

The proposed functional form appears to be a worthy addition to the existing class of single parameter Lorenz curves. The new specification is shown to meet the required regularity conditions for a Lorenz curve and demonstrates a strong capacity for modeling income data. The ability for this Lorenz curve to effectively model data is likely to be due a similarity of shapes behind the underlying PDF and typical income distributions.

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